JOURNAL OF APPROXIMATION THEORY 27, 99–112 (1979)

A Class of Generalized Cardinal Splines

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Communicated by Richard S. Varga

Received January 12, 1978

1. INTRODUCTION

In [6, 7], Schoenberg developed an elegant theory of cardinal interpolation by cardinal splines which was later extended to cardinal Hermite interpolation by Lipow and Schoenberg [3] and by Lee *et al.* [2]. More recently Tzimbalario [9] and Mohapatra and Sharma [4] have derived analogous results for certain classes of cardinal discrete splines. In this paper we derive results for a broad class of generalized cardinal splines which both unify and generalize the results of all the above papers. In particular we extend the results of [9] to cardinal Hermite interpolation.

Let P_n denote the space of all real-valued polynomials of degree not exceeding n, and for $1 \leqslant s \leqslant n$, let $\Gamma = (\gamma_0, \gamma_1, ..., \gamma_{n-s})$ denote a set of linearly independent linear functionals on P_n. We define $\mathscr{S}_n(\Gamma)$ to be the class of all functions S from $\mathbb{R}$ to itself such that for $\nu = 0, \pm 1, \pm 2, ..., S \mid [\nu, \nu + 1) = S_\nu \in P_n$ and $\gamma[S_{\nu-1}(x + \nu)] = \gamma[S_\nu(x + \nu)]$, $\forall \gamma \in \Gamma$.

If $\gamma_i(p) = p^{(i)}(0)$, $i = 0, ..., n - s$, then $\mathscr{S}_n(\Gamma)$ is the class of cardinal splines with integer nodes of multiplicity s studied in [2, 3]. If $\gamma_i(p) = p^{(i)}(0)$, $i = 0, ..., n - s - 1$, and $\gamma_{n-s}(p) = p^{(n-s+1)}(0)$, then $\mathscr{S}_n(\Gamma)$ consists of the cardinal g-splines studied by Lee and Sharma [1]. The cardinal discrete splines of [4, 9] are obtained by putting $s = 1$, $\gamma_i(p) = p(ih)$ and $\gamma_i(p) = p^{(i)}(ih)$, $i = 0, 1, ..., n - 1$ $(0 < h < 1/n)$.

We note that our above definition of $\mathscr{S}_n(\Gamma)$ is the analogy for cardinal polynomial splines of Schumakers' classes of generalized splines as defined in [8].

Now if $\mathscr{P} = \{p_1, p_2, ..., p_s\}$ is a basis for $\{p \in P_n : \gamma(p) = 0, \forall \gamma \in \Gamma\}$, then it is easily seen that $\mathscr{S}_n(\Gamma)$ comprises all functions S of the form

$$S(x) = P(x) + \sum_{k=1}^{s} c_1^{(k)} p_k(x-1)_+ + \sum_{k=1}^{s} c_2^{(k)} p_k(x-2)_+ + \cdots + \sum_{k=1}^{s} c_0^{(k)} p_k(x)_- + \sum_{k=1}^{s} c_{-1}^{(k)} p_k(x+1)_- + \cdots, \tag{1.1}$$

0021-9045/79/100099–14$02.00/0

where $P \in P_n$ and

$$\begin{aligned} p(x)_+ &= p(x) \quad &&\text{if} \quad x \geqslant 0 \\ &= 0 \quad &&\text{if} \quad x < 0 \end{aligned} \tag{1.2}$$

$$\begin{aligned} p(x)_- &= 0 \quad &&\text{if} \quad x \geqslant 0 \\ &= p(x) \quad &&\text{if} \quad x < 0. \end{aligned} \tag{1.3}$$

We shall alternatively write $\mathscr{S}_n(\Gamma)$ as $\mathscr{S}_n(\mathscr{P})$.

Then $\mathscr{P} = \{x^{n-s+1}, x^{n-s+2},..., x^n\}$ gives the cardinal splines of [2, 3] and $\mathscr{P} = \{x^{n-s}, x^{n-s+2}, x^{n-s+3},..., x^n\}$ gives the cardinal g-splines of [1], while for the cardinal discrete splines of [9, 4] $\mathscr{P}$ comprises the single polynomial $x^{(n)} = x(x-h)(x-2h) \cdots (x-(n-1)h)$ and $G_n(x) = x(x-nh)^{n-1}$, respectively.

Our main result involves the class of polynomials $\mathscr{P} = \{p, xp,..., x^{s-1}p\}$, where p is a polynomial of exact degree $n - s + 1$ whose zeros lie in the intersection of the circles $|z - z_0| < |z_0|$ and $|z - \bar{z}_0| < |\bar{z}_0|$, where

$$z_0 = \frac{ie^{i\pi/(s+1)}}{2\sin(\pi/(s+1))} + \alpha,$$

and perhaps with zeros of multiplicity m at α and of multiplicity l at $\alpha - 1$, where α is a number in $[0, 1)$. Suppose α is not a zero of the polynomial $\Pi_n(p, xp, .., x^{s-1}p\colon (-1)^s)$ in x defined in (2.4). Then our main result is

THEOREM 1. *Given s bi-infinite sequences of data $y^{(\rho)} = (y_\nu^{(\rho)})$ $(\rho = 0, 1,..., s-1)$ satisfying*

$$y_\nu^{(\rho)} = O(|\nu|^\gamma) \qquad (\rho = 0, 1,..., s-1) \tag{1.4}$$

for some $\gamma > 0$, there exists a unique spline function $S \in \mathscr{S}_n(\mathscr{P})$ satisfying

$$S^{(\rho)}((\nu + \alpha)+) = y_\nu^{(\rho)} \qquad (\rho = 0, 1,..., s-1) \quad \forall \text{ integer } \nu, \tag{1.5}$$

such that

$$S(x) = O(|x|^\gamma). \tag{1.6}$$

Our approach in solving the above interpolation problem is the same as in [2, 3]. In Section 2 we study the eigensplines and the zeros of the polynomials $\Pi_{n,r}(\mathscr{P}:\lambda)$ as a polynomial in λ, while a sketch of the proof for Theorem 1 is given in Section 3.

2. EIGENSPLINES

Let $0 \leqslant \alpha < 1$ and set

$$\mathscr{S}_n^{\alpha}(\mathscr{P}) = \mathscr{S}_n^{\alpha}(p_1, p_2, \ldots, p_s) = \{S \in \mathscr{S}_n(\mathscr{P}) : S^{(\rho)}((\nu + \alpha)+) = 0 \quad \forall \rho = 0, 1, \ldots, s-1 \text{ and } \forall \text{ integers } \nu\}. \tag{2.1}$$

A function $S \in \mathscr{S}_n^{\alpha}(\mathscr{P})$, $S \neq 0$, is called an *eigenspline* if it satisfies the functional relation

$$S(x+1) = \lambda S(x), \qquad \forall x \in \mathbb{R} \text{ and for some } \lambda \neq 0. \tag{2.2}$$

The number λ is called the *eigenvalue* of S.

Next we set out to determine the eigensplines in $\mathscr{S}_n^{\alpha}(\mathscr{P})$. Let S be an eigenspline in $\mathscr{S}_n^{\alpha}(\mathscr{P})$ with eigenvalue λ, and let P be the polynomial component of S in the interval $[0, 1)$. Then $P^{(\rho)}(\alpha) = 0$ for $\rho = 0, 1, \ldots, s-1$, and we can write

$$P(x) = a_s(x-\alpha)^s + a_{s+1}(x-\alpha)^{s+1} + \cdots + a_n(x-\alpha)^n.$$

From the relation $S(x+1) = \lambda S(x)$, $\forall x \in \mathbb{R}$, and (1.1) we have

$$P(x+1) - \lambda P(x) = \sum_{k=1}^{s} c_1^{(k)} p_k(x).$$

Hence

$$P^{(\rho)}(\alpha+1) - \lambda P^{(\rho)}(\alpha) = \sum_{k=1}^{s} c_1^{(k)} p_k^{(\rho)}(\alpha), \qquad \forall \rho = 0, 1, \ldots, n. \tag{2.3}$$

Let $p_k(x) = a_{k0} + a_{k1}(x-\alpha) + a_{k2}(x-\alpha)^2 + \cdots + a_{kn}(x-\alpha)^n$, and writing (2.3) in increasing ρ from 0 to n, we have a system of $(n+1)$ equations in $(n+1)$ unknowns $a_s, a_{s+1}, \ldots, a_n, c_1^{(1)}, c_1^{(2)}, \ldots, c_1^{(s)}$.

The matrix of the system is

$$
\left\| \begin{array}{ccccc}
1 & \binom{s}{1} & 2!\binom{s}{2} & \cdots & (s-1)!\binom{s}{s-1} \\
1 & \binom{s+1}{1} & 2!\binom{s+1}{2} & \cdots & (s-1)!\binom{s+1}{s-1} \\
1 & \binom{s+1}{1} & 2!\binom{s+1}{2} & & \cdot \\
\vdots & \vdots & \vdots & & \vdots \\
1 & \binom{n-1}{1} & 2!\binom{n-1}{2} & \cdots & (s-1)!\binom{n-1}{s-1} \\
1 & \binom{n}{1} & 2!\binom{n}{2} & \cdots & (s-1)!\binom{n}{s-1} \\
a_{10} & 1!\,a_{11} & 2!\,a_{12} & \cdots & (s-1)!\,a_{1s-1} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{s-10} & 1!\,a_{s-11} & 2!\,a_{s-12} & \cdots & (s-1)!\,a_{s-1s-1} \\
c_{s0} & 1!\,a_{s1} & 2!\,a_{s2} & \cdots & (s-1)!\,a_{ss-1}
\end{array} \right.
$$

$$
\left. \begin{array}{cccc}
s!\,(1-\lambda) & 0 & & 0 \\
s!\binom{s+1}{s} & (s+1)!\,(1-\lambda) & & \cdot \\
\cdot & & & \cdot \\
\vdots & & & \vdots \\
s!\binom{n-1}{s} & \cdots & (n-1)!\,(1-\lambda) & 0 \\
s!\binom{n}{s} & \cdots & (n-1)!\binom{n}{n-1} & n!\,(1-\lambda) \\
s!\,a_{1s} & \cdots & (n-1)!\,a_{1n-1} & n!\,a_{1n} \\
\vdots & & & \vdots \\
s!\,a_{s-1s} & \cdots & (n-1)!\,a_{s-1n-1} & n!\,a_{s-1n} \\
s!\,a_{ss} & \cdots & (n-1)!\,a_{sn-1} & n!\,a_{sn}
\end{array} \right\|
$$

Therefore, in order that P be nontrivial, the determinant

$$\Pi_n(\mathscr{P} : \lambda) \equiv \Pi_n(p_1, p_2, \ldots, p_s : \lambda)$$

$$= \begin{vmatrix} 1 & \binom{s}{1} & \binom{s}{2} & \cdots & \binom{s}{s-1} & (1-\lambda) & 0 & & \cdots & 0 \\ 1 & \binom{s+1}{1} & \binom{s+1}{2} & \cdots & \binom{s+1}{s-1} & \binom{s+1}{s} & (1-\lambda) & 0 & \cdots & \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & & \vdots \\ 1 & \binom{n-1}{1} & \binom{n-1}{2} & \cdots & \binom{n-1}{s-1} & \binom{n-1}{s} & \cdots & (1-\lambda) & & 0 \\ 1 & \binom{n}{1} & \binom{n}{2} & \cdots & \binom{n}{s-1} & \binom{n}{s} & \cdots & \binom{n}{n-1} & & (1-\lambda) \\ a_{10} & a_{11} & a_{12} & \cdots & a_{1s-1} & a_{1s} & \cdots & a_{1n-1} & & a_{1n} \\ a_{20} & a_{21} & a_{22} & \cdots & a_{2s-1} & a_{2s} & \cdots & a_{2n-1} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & & \vdots \\ a_{s-10} & a_{s-11} & a_{s-12} & \cdots & a_{s-1s-1} & a_{s-1s} & \cdots & a_{s-1n-1} & & a_{s-1n} \\ a_{s0} & a_{s1} & a_{s2} & \cdots & a_{ss-1} & a_{ss} & \cdots & a_{sn-1} & & a_{sn} \end{vmatrix} \tag{2.4}$$

must be zero.

Observe that if $p_k(x) = x^{n-s+k}$ for $k = 1, 2, \ldots, s$, then the polynomial $\Pi_n(\mathscr{P}; \lambda)$ reduces to $\Pi_n(\alpha; \lambda)$ of [2].

Now for $s \leqslant r \leqslant n+1$, let $\Pi_{n,r}(\mathscr{P} : \lambda) \equiv \Pi_{n,r}(p_1, p_2, \ldots, p_s : \lambda)$ be the determinant obtained from $\Pi_n(\mathscr{P} : \lambda)$ by deleting the first $(r-s)$ rows and the last $(r-s)$ columns. We shall put $\Pi_{n,r}(p_1, p_2, \ldots, p_s : \lambda) = 0$ for $r < s$ and $r > n+1$. Following [3] we let $\Pi_{n,r}(\lambda)$ denote the determinant of the matrix obtained by deleting the first r rows and the last r columns of $\|\binom{i}{j} - \lambda\delta ij\|$, $i, j = 0, 1, \ldots, n$. We shall also write $\Pi_{n,r}(\mathscr{P}) \equiv \Pi_{n,r}(p_1, p_2, \ldots, p_s) = \Pi_{n,r}(\mathscr{P} : 0)$.

LEMMA 2.1. *Let n, r, s be positive integers with $n \geqslant r \geqslant s \geqslant 1$. Then the following identities hold*:

$$r\Pi_{n,r+1}(p : \lambda)\, \Pi_{n-1,r-1}(\lambda) = \Pi_{n-1,r}(p' : \lambda)\, \Pi_{n,r}(\lambda) - (n-r+1)\, \Pi_{n,r}(p : \lambda)\, \Pi_{n-1,r}(\lambda). \tag{2.5}$$

$$\begin{aligned} \Pi_{n,r}(p_1, p_2, \ldots, p_s : \lambda)\, \Pi_{n,r-1}(p_2, p_3, \ldots, p_{s-1} : \lambda) &= \Pi_{n,r}(p_1, p_2, \ldots, p_{s-1} : \lambda)\, \Pi_{n,r-1}(p_2, p_3, \ldots, p_s : \lambda) \\ &\quad - \Pi_{n,r}(p_2, p_3, \ldots, p_s : \lambda)\, \Pi_{n,r-1}(p_1, p_2, \ldots, p_{s-1} : \lambda). \end{aligned} \tag{2.6}$$

$$\begin{aligned} r\Pi_{n,r+1}(p_1, p_2, \ldots, p_s : \lambda)\, \Pi_{n-1,r-1}(p_1', \ldots, p_{s-1}' : \lambda) &= \Pi_{n-1,r}(p_1', \ldots, p_s' : \lambda)\, \Pi_{n,r}(p_1, \ldots, p_{s-1} : \lambda) \\ &\quad - (n-r+s)\, \Pi_{n,r}(p_1, \ldots, p_s : \lambda)\, \Pi_{n-1,r}(p_1', \ldots, p_{s-1}' : \lambda). \end{aligned} \tag{2.7}$$

Proof. The proof of the above identities involves the same method as that used in [1], employing Karlin's identity. ∎

LEMMA 2.2. *Suppose* $p(x) = \sum_{i=0}^{n} a_i x^i$. *Then for* $1 \leqslant r \leqslant n + 1$,

$$\Pi_{n,r}(p) = a_{n-r+1} - \binom{r}{1} a_{n-r} + \binom{r+1}{2} a_{n-r+1} + \cdots + (-1)^{n-r+1} \binom{n}{n-r+1} a_0 . \tag{2.8}$$

Proof. The result is obtained by expanding the determinant representing $\Pi_{n,r}(p)$ along the last row. ∎

LEMMA 2.3. *For any polynomial* p,

$$(1 - x)^n \, p\left(\frac{x}{1 - x}\right) = \sum_{i=0}^{n} \Pi_{n,n+1-i}(p) \, x^i. \tag{2.9}$$

Proof. Let $p(x) = \sum_{i=0}^{n} a_i x^i$. *Then*

$$\begin{aligned} (1 - x)^n \, p\left(\frac{x}{1 - x}\right) &= \sum_{i=0}^{n} a_i x^i (1 - x)^{n-i} \\ &= \sum_{i=0}^{n} \sum_{j=i}^{n} a_i (-1)^{j-i} \binom{n-i}{j-i} x^j. \end{aligned}$$

Interchanging the order of summation and applying Lemma 2.2, the result (2.9) follows. ∎

LEMMA 2.4. *If* $n + 1 \geqslant r$ *and* $a_n = 0$, *then*

$$\Pi_{n,r}(p) = \Pi_{n-1,r-1}(p) - \Pi_{n-1,r}(p) \tag{2.10}$$

and

$$\Pi_{n,r}(xp) = \Pi_{n-1,r}(p). \tag{2.11}$$

Relations (2.10) and (2.11) follow from (2.9) if $a_n = 0$. ∎

Now for $n + 1 \geqslant r \geqslant s \geqslant 1$, let $\Delta_{n,r}(p_1, p_2, \ldots, p_s : \lambda)$ denote the determinant

$$\begin{vmatrix} \Pi_{n,r}(p_1 : \lambda) & \Pi_{n,r-1}(p_1 : \lambda) & \cdots & \Pi_{n,r-s+1}(p_1 : \lambda) \\ \Pi_{n,r}(p_2 : \lambda) & \Pi_{n,r-1}(p_2 : \lambda) & \cdots & \Pi_{n,r-s+1}(p_2 : \lambda) \\ \vdots & \vdots & & \vdots \\ \Pi_{n,r}(p_s : \lambda) & \Pi_{n,r-1}(p_s : \lambda) & \cdots & \Pi_{n,r-s+1}(p_s : \lambda) \end{vmatrix}$$

LEMMA 2.5. *If* $n + 1 \geqslant r \geqslant s \geqslant 1$, *then*

$$\begin{aligned} &\Pi_{n,r}(p_1, p_2, \ldots, p_s : \lambda)\, \Pi_{n,r-1}(\lambda)\, \Pi_{n,r-2}(\lambda) \cdots \Pi_{n,r-s+1}(\lambda) \\ &\quad = \Delta_{n,r}(p_1, p_2, \ldots, p_s : \lambda). \end{aligned} \tag{2.12}$$

Proof. Our proof is by induction on s. Clearly the identity (2.12) is true for $s = 1$, and for all $n \geqslant r$. Suppose that it is true for $s = k$. By Sylvester's identity we have

$$\begin{aligned} &\Delta_{n,r}(p_1, p_2, \ldots, p_{k+1} : \lambda)\, \Delta_{n,r-1}(p_2, p_3, \ldots, p_k : \lambda) \\ &\quad = \Delta_{n,r-1}(p_2, p_3, \ldots, p_{k+1} : \lambda)\, \Delta_{n,r}(p_1, p_2, \ldots, p_k : \lambda) \\ &\qquad - \Delta_{n,r}(p_2, p_3, \ldots, p_{k+1} : \lambda)\, \Delta_{n,r-1}(p_1, p_2, \ldots, p_k : \lambda). \end{aligned} \tag{2.13}$$

Hence from (2.6), (2.13), and the induction hypothesis, the assertion is true for $s = k + 1$, and by induction it is true for all $r \geqslant s \geqslant 1$. ∎

LEMMA 2.6. *If p is a polynomial of exact degree* $(n - s + 1)$, *then*

$$\Pi_{n,r}(p, xp, \ldots, x^{s-1}p) = \begin{vmatrix} \Pi_{n-s+1,r-s+1}(p) & \Pi_{n-s+1,r-s}(p) & \cdots & \Pi_{n-s+1,r-2s+2}(p) \\ \Pi_{n-s+1,r-s+2}(p) & \Pi_{n-s+1,r-s+1}(p) & \cdots & \Pi_{n-s+1,r-2s+3}(p) \\ \vdots & \vdots & & \vdots \\ \Pi_{n-s+1,r}(p) & \Pi_{n-s+1,r-1}(p) & \cdots & \Pi_{n-s+1,r-s+1}(p) \end{vmatrix} \tag{2.14}$$

Proof. The assertion follows from (2.12), using (2.11) and (2.10). ∎

Next we require a lemma on k-positive sequences.

LEMMA 2.7. *Let k be a positive integer and suppose the polynomial* $a_0 + a_1 z + \cdots + a_m z^m$ $(a_m > 0)$ *has no roots in the sector* $|\arg z| < k\pi/(k + 1)$. *Let* $a_i = 0$ *for* $i < 0$ *and* $i > m$. *Then every minor of order* $\leqslant k$ *of the matrix* $\| a_{j-i} \|$ *is strictly positive unless it contains a zero row or column. Moreover the constant* $k\pi/(k + 1)$ *is the best possible.*

Proof. The result follows almost immediately from the work of Schoenberg [5]. The result is clearly true for $m = 1$. That it is true for $m = 2$ when the polynomial has complex roots follows from Theorem 3 in [5]. It then follows that the assertion is true for all m, since the class of all k- positive sequences is closed with respect to the operation of convolution of sequences. That the constant $k\pi/(k+1)$ is the best possible follows by putting $m = 2$ in Theorem 1 of [5]. ▮

By a suitable translation we may assume without loss of generality that $\alpha = 0$.

LEMMA 2.8. *If p has exact degree $(n - s + 1)$, $p(0) > 0$, and all roots lie in the intersection of the circles $|z - z_0| < |z_0|$ and $|z - \bar{z}_0| < |\bar{z}_0|$, where*

$$z_0 = \frac{ie^{i\pi/(s+1)}}{2\sin(\pi/(s+1))},$$

then

$$\Pi_{n,r}(p, xp, ..., x^{s-1}p : \lambda) = b\lambda^{n-2r+s+1} + \cdots + a,$$

where $a > 0$ and sign $b = (-1)^{(r+1)(n+s+1)}$.

Proof. By Lemmas 2.3, 2.6, and 2.7, it follows that $\Pi_{n,r}(p, xp, ..., x^{s-1}p) > 0$, $\forall r = s, s+1, ..., n+1$. Since

$$\Pi_{n,r}(p, xp, ..., x^{s-1}p : \lambda) = (-1)^{(n+s+1)(r+1)}\Pi_{n,n-r+s+1}(p, xp, ..., x^{s-1}p)\,\lambda^{n-2r+s+1} + \cdots + \Pi_{n,r}(p, xp, ..., x^{s-1}p),$$

the result follows. ▮

THEOREM 2. *Let n, r, s be positive integers such that $s \leqslant r \leqslant \frac{1}{2}(n+s+1)$, and suppose that p has exact degree $(n - s + 1)$ and all its roots lie in the intersection of the circles $|z - z_0| < |z_0|$ and $|z - \bar{z}_0| < |\bar{z}_0|$, where*

$$z_0 = \frac{ie^{i\pi/(s+1)}}{2\sin(\pi/(s+1))}.$$

Then for $s = 1$, $\Pi_{n,r}(p : \lambda)$ has $(n - 2r + 2)$ distinct real zeros of sign $(-1)^r$ interlacing with the zeros of $\Pi_{n,r}(\lambda)$ and of $\Pi_{n-1,r}(p' : \lambda)$, and for $s > 1$, $\Pi_{n,r}(p, xp, ..., x^{s-1}p : \lambda)$ has $(n - 2r + s + 1)$ distinct real zeros of sign $(-1)^r$ interlacing with the zeros of $\Pi_{n,r}(xp, x^2p, ..., x^{s-1}p : \lambda)$ and of $\Pi_{n-1,r}(p', (xp)', ..., (x^{s-1}p)' : \lambda)$.

Proof. If n is even and $r = \frac{1}{2}(n+2)$, then $\Pi_{n,r}(p : \lambda)$ is a positive constant. If n is odd and $r = \frac{1}{2}(n+1)$, then $\Pi_{n,r}(p : \lambda) = b\lambda + a$, where $a > 0$ and sign $b = (-1)^{r+1}$, so that the zero of $\Pi_{n,r}(p : \lambda)$ is of sign $(-1)^r$. Now take $k < \frac{1}{2}(n+1)$ and suppose $\Pi_{n,r}(p : \lambda)$ has distinct real zeros of sign $(-1)^r$ for $r \geqslant k+1$. We shall show that $\Pi_{n,k}(p : \lambda)$ has distinct real zeros of sign $(-1)^k$. Evaluating (2.5) at the zeros of $\Pi_{n,k}(\lambda)$ and using the fact that $\Pi_{n,k}(\lambda)$ and $\Pi_{n-1,k}(\lambda)$ have interlacing zeros of sign $(-1)^k$, it follows that $\Pi_{n,k}(p : \lambda)$ has distinct real zeros of sign $(-1)^r$ interlacing with the zeros of $\Pi_{n,k}(\lambda)$. Then evaluating (2.5) at the zeros of $\Pi_{n,k}(p : \lambda)$, we see that $\Pi_{n-1,k}(p' : \lambda)$ has distinct real zeros of sign $(-1)^k$ which interlace with the zeros of $\Pi_{n,k}(p : \lambda)$. Hence the result for $s = 1$ is proved by downward induction on r.

Next, take $\rho > 1$ and suppose the assertion is true for $s \leqslant \rho - 1$. We want to show that it is true for $s = \rho$. If $(n + \rho)$ is odd, the result is true for $r = \frac{1}{2}(n + \rho + 1)$. If $(n + s)$ is even, the result is true for $r = \frac{1}{2}(n + s)$.

Now take $k < \frac{1}{2}(n + \rho)$, and suppose $\Pi_{n,r}(p, xp,..., x^{\rho-1}p : \lambda)$ has distinct real zeros of sign $(-1)^r$ for $r \geqslant k + 1$. From (2.7),

$$\begin{aligned} &k\Pi_{n,k+1}(p, xp,..., x^{\rho-1}p : \lambda)\, \Pi_{n-1,k-1}((xp)', (x^2p)',..., (x^{\rho-1}p)' : \lambda) \\ &= \Pi_{n-1,k}(p', (xp)',..., (x^{\rho-1}p)' : \lambda)\, \Pi_{n,k}(xp, x^2p,..., x^{\rho-1}p : \lambda) \\ &\quad - (n - r + \rho)\, \Pi_{n,k}(p, xp,..., x^{\rho-1}p : \lambda)\, \Pi_{n-1,k}((xp)', (x^2p)',..., (x^{\rho-1}p)' : \lambda). \end{aligned}$$

Evaluating this at the zeros of $\Pi_{n,k}(xp, x^2p,..., x^{\rho-1}p : \lambda)$ and using the induction hypothesis we see that $\Pi_{n,k}(p, xp,..., x^{\rho-1}p : \lambda)$ has real distinct zeros of sign $(-1)^r$ which interlace with the zeros of $\Pi_{n,k}(xp, x^2p,..., x^{\rho-1}p : \lambda)$. Then evaluating the same relation at the zeros of $\Pi_{n,k}(p, xp,..., x^{\rho-1}p : \lambda)$, we see that $\Pi_{n-1,k}(p', (xp)',..., (x^{\rho-1}p)' : \lambda)$ has distinct real zeros of sign $(-1)^r$ which interlace with the zeros of $\Pi_{n,r}(p, xp,..., x^{\rho-1}p : \lambda)$.

The result follows by induction. ∎

We next consider the possibility that the polynomial p has zeros at α or $\alpha - 1$. We again assume $\alpha = 0$.

LEMMA 2.9. *If the polynomial* $p(x) = \sum_{i=0}^{n} a_i x^i$ *has a zero at* $x = 0$ *of multiplicity* m, *then for* $1 \leqslant r \leqslant m, r \leqslant \frac{1}{2}(n + 1)$, $r \leqslant n - m + 1, \Pi_{n,r}(p : \lambda)$ *has degree* $(n - 2r + 1)$ *and the coefficient of* λ^{n-2r+1} *is* $(-1)^{n(r+1)} \Pi_{n,n-r+2}(\tilde{p})$, *where* $p(x) = \tilde{p}(x + 1)$.

Proof. By (2.9),

$$\sum_{i=0}^{n} \Pi_{n,n-1-i}(\tilde{p})\, x^i = (1 - x)^n\, \tilde{p}\left(\frac{x}{1 - x}\right) = (1 - x)^n\, p\left(\frac{1}{1 - x}\right).$$

So

$$\Pi_{n,n+1-i}(\tilde{p}) = (-1)^i \sum_{j=0}^{n-i} a_j \binom{n - j}{i}.$$

By expanding $\Pi_{n,r}(p : \lambda)$, we see that $\Pi_{n,r}(p : \lambda)$ has degree $(n - 2r + 1)$ and the coefficient of λ^{n-2r+1} is $\sum_{j=r}^{n-s+1} a_j \binom{n-j}{r-1}$. Since $a_0 = \cdots = a_{r-1} = 0$, the result follows. ∎

LEMMA 2.10. *Suppose the polynomial* $p(x) = \sum_{i=0}^{n-s+1} a_i x^i$ *has a zero at* $x = 0$ *of multiplicity* m. *Then for* $m \leqslant r < m + s$, $1 \leqslant s \leqslant r \leqslant n - m + 1$, $\Pi_{n,r}(p, xp,..., x^{s-1}p : \lambda)$ *has degree* $(n - r - m + 1)$ *and the coefficient of* $\lambda^{n-r-m+1}$ *is:*

$$(-1)^{n(r+1)+(s+1)(m+1)} a_m^{r-m} \times \begin{vmatrix} \Pi_{N,n-m+2}(q) & \Pi_{N,n-m+1}(q) & \cdots & \Pi_{N,n-2m+r-s+3}(q) \\ \Pi_{N,n-m+3}(q) & \Pi_{N,n-m+2}(q) & \cdots & \Pi_{N,n-2m+r-s+1}(q) \\ \vdots & \vdots & & \vdots \\ \Pi_{N,n-r+s+1} & \Pi_{N,n-r+s}(q) & \cdots & \Pi_{N,n-m+2}(q) \end{vmatrix} \tag{2.16}$$

where $N = n - s + r - m + 1$ *and* $q(x) = (x + 1)^{r-m} p(x + 1)$.

For $s \leqslant r \leqslant m$, $r \leqslant \frac{1}{2}(n + 1)$, and $r \leqslant n - m + 1$, the degree of $\Pi_{n,r}(p, xp,..., x^{s-1}p : \lambda)$ is $(n - 2r + 1)$ and the coefficient is (2.16) with m replaced by r.

Proof. The result follows from Lemmas 2.5, 2.9, and 2.6. ∎

By a similar method, we have the following.

LEMMA 2.11. *Suppose the polynomial* p *of degree* $(n - s + 1)$ *has a zero at* $x = -1$ *of multiplicity* l. *Then for* $l \leqslant r < l + s$, $1 \leqslant s \leqslant r \leqslant n - l + 1$, $\Pi_{n,r}(p, xp,..., x^{s-1}p : \lambda)$ *has a zero at* $\lambda = 0$ *of multiplicity* $(s + l - r)$ *and the coefficient of* λ^{s+l-r} *is*

$$(-1)^{(r+l)(s+1)} \Pi_{n-s+1,l+1}(p) \Pi_{n-s+1,l+2}(p) \cdots \Pi_{n-s+1,r}(p) \times \begin{vmatrix} \Pi_{N,r-s+1}(Q) & \Pi_{N,r-s}(Q) & \cdots & \Pi_{N,2r-2s-l+2}(Q) \\ \Pi_{N,r-s+2}(Q) & \Pi_{N,r-s+1}(Q) & \cdots & \Pi_{N,2r-2s-l+3}(Q) \\ \vdots & & & \\ \Pi_{N,l}(Q) & \Pi_{N,l-1}(Q) & \cdots & \Pi_{N,r-s+1}(Q) \end{vmatrix},$$

where $N = n - s + r - l + 1$ *and* $Q(x) = (x - 1)^{r-l} p(x - 1)$.

For $s \leqslant r \leqslant l$, and $r \leqslant \frac{1}{2}(n + 1)$, $r \leqslant n - l + 1$, $\Pi_{n,r}(p, xp,..., x^{s-1}p : \lambda)$ has a root at $\lambda = 0$ of multiplicity s and the coefficient of λ^s is (2.17) with l replaced by r.

LEMMA 2.12. *Suppose the polynomial* $p(x) = \sum_{i=0}^{n-s+1} a_i x^i$, $a_{n-s+1} > 0$, *has a zero at* $x = -1$ *of multiplicity* l *and a zero at* $x = 0$ *of multiplicity* m, *and all other zeros lie in the intersection of the circles* $| z - z_0 | < | z_0 |$ *and* $| z - \bar{z}_0 | < | \bar{z}_0 |$, *where*

$$z_0 = \frac{ie^{i\pi/(s+1)}}{2 \sin(\pi/(s + 1))}.$$

Let

$$\begin{aligned} \alpha &= n - 2r + s + 1 && (r \geqslant m + s) \\ &= n - r - m + 1 && (m \leqslant r < m + s) \\ &= n - 2r + 1 && (s \leqslant r < m) \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} \beta &= 0 && (r \geqslant l + s) \\ &= s + l - r && (l \leqslant r \leqslant l + s) \\ &= s && (s \leqslant r \leqslant l). \end{aligned} \tag{2.19}$$

Then if $\alpha \geqslant \beta$, $r \leqslant n - m + 1$, $r \leqslant n - l + 1$, $\Pi_{n,r}(p, xp,..., x^{s-1}p : \lambda) = C_\alpha \lambda^\alpha + C_{\alpha-1}\lambda^{\alpha-1} + \cdots + C_\beta \lambda^\beta$, *where* $\operatorname{sgn} C_\alpha = (-1)^{(r+1)\alpha}$ *and* $\operatorname{sgn} C^\beta = (-1)^{(r+1)\beta}$.

Proof. It follows from Lemma 2.7 that if k is a positive integer and the polynomial $b_0 + b_1 z + \cdots + b_m z^m$ $(b_m > 0)$ has no roots in the sector $|\arg(-z)| < k\pi/(k+1)$, then every minor of the matrix $\|(-1)^{j-i-m}\, b_{j-i}\|$ is strictly positive unless it contains a zero row or column (where $b_i = 0$ for $i < 0$ and $i > m$). The required result then follows from Lemmas 2.3, 2.10, and 2.11. ∎

THEOREM 3. *Let* $p, n, r, s, l, m, \alpha, \beta$ *be as in Lemma* 2.12, *and* $\alpha \geqslant \beta$, $r \leqslant n - m + 1$, $r \leqslant n - l + 1$. *Then* $\Pi_{n,r}(p, xp,..., x^{s-1}p : \lambda)$ *has* $\alpha - \beta$ *distinct real zeros of sign* $(-1)^r$. *For* $s = 1$, *these zeros interlace with the zeros of* $\Pi_{n,r}(\lambda)$ *and of* $\Pi_{n-1,r}(p' : \lambda)$, *and for* $s > 1$ *they interlace with the zeros of* $\Pi_{n,r}(xp, x^2p,..., x^{s-1}p : \lambda)$ *and of* $\Pi_{n-1,r}(p', (xp)',..., (x^{s-1}p)' : \lambda)$

Proof. This follows exactly the same lines as the proof of Theorem 1, applying Lemma 2.12. ∎

For the rest of this paper we shall assume that p is a polynomial of exact degree $(n - s + 1)$ whose zeros lie in the intersection of the circles $|z - z_0| < |z_0|$ and $|z - \bar{z}_0| < |\bar{z}_0|$, where

$$z_0 = \frac{i e^{i\pi/(s+1)}}{2 \sin(\pi/(s+1))} + \alpha,$$

and perhaps with zeros of multiplicity m at α and of multiplicity l at $\alpha - 1$. Let $\mathscr{P} = \{p, xp,..., x^{s-1}p\}$. Then the polynomial $\Pi_n(\mathscr{P} : \lambda) \equiv \Pi_{n,s}(\mathscr{P} : \lambda)$ has d distinct zeros, where d is given by

$$\begin{aligned} d &= n - s - l - m + 1 && (m \leqslant s,\ l \leqslant s) \\ &= n - 2s - m + 1 && (m \leqslant s,\ l > s) \\ &= n - 2s - l + 1 && (m > s,\ l \leqslant s) \\ &= n - 3s + 1 && (m > s,\ l > s). \end{aligned} \tag{2.20}$$

Let the zeros of $\Pi_n(\mathscr{P} : \lambda)$ be $\lambda_1, \lambda_2, \ldots, \lambda_d$. For each λ_i $(i = 1, 2, \ldots, d)$ let $P_i(x)$ be the polynomial corresponding to a solution of the system of equations (2.3) with $\lambda = \lambda_i$, and define $S_i \in \mathscr{S}_n^\alpha(\mathscr{P})$ such that $S_i(x) = P_i(x)$, $\forall x \in [0, 1)$, and $S_i(x + 1) = \lambda_i S_i(x)$, $\forall x \in \mathbb{R}$. Since λ_i are distinct the functions $\{S_1, S_2, \ldots, S_d\}$ are linearly independent. Furthermore using the same argument as in [3] it can be shown that the dimension of $\mathscr{S}_n^\alpha(\mathscr{P})$ is d. More precisely we have

LEMMA 2.13. *Let p be a polynomial of exact degree $(n - s + 1)$ whose zeros lie in the intersection of the circles $|z - z_0| < |z_0|$ and $|z - \bar{z}_0| < |\bar{z}_0|$, and perhaps with zeros of multiplicity m at α and of multiplicity l at $\alpha - 1$. Then the dimension of $S_n^\alpha(\mathscr{P})$ is d, where d is given by* (2.20).

Now, since the eigensplines $\{S_1, S_2, \ldots, S_d\}$ are linearly independent, it follows from Lemma 2.13 that they form a basis for $S_n^\alpha(\mathscr{P})$. Thus we have

LEMMA 2.14. *Every $S \in \mathscr{S}_n^\alpha(\mathscr{P})$ has a unique representation of the form*

$$S(x) = \sum_{i=1}^{d} c_i S_i(x).$$

3. PROOF OF THEOREM 1

The proof follows the same pattern as [2] and we shall give only a sketch. Since α is not a zero of $\Pi_n(p, xp, \ldots, x^{s-1}p : (-1)^s)$, none of the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_d$ lie on the unit circle. Suppose

$$\begin{aligned} |\lambda_i| &< 1 \qquad \text{for} \quad i = 1, 2, \ldots, k, \\ |\lambda_i| &> 1 \qquad \text{for} \quad i = k + 1, k + 2, \ldots, d, \end{aligned} \tag{3.1}$$

where $0 \leqslant k \leqslant d$. For $\rho = 0, 1, \ldots, s - 1$, let

$$L_\rho(x) \begin{cases} P(x) & (0 \leqslant x < 1) \\ \sum_{i=1}^{k} c_i S_i(x) & (x \geqslant 1) \\ \sum_{i=k+1}^{d} c_i S_i(x) & (x < 0), \end{cases} \tag{3.2}$$

where

$$P(x) = \frac{(x - \alpha)^\rho}{\rho!} + a_s(x - \alpha)^s + a_{s+1}(x - \alpha)^{s+1} + \cdots + a_n(x - \alpha)^n. \tag{3.3}$$

Let $P_1(x) = \sum_{i=1}^{k} c_i S_i(x)$ for $x \in [1, 2)$ and $P_{-1}(x) = \sum_{i=k+1}^{d} c_i S_i(x)$ for $x \in [-1, 0)$. Suppose the zeros of $p(x)$ are $x_1, x_2, \ldots, x_{n-s+1}$. Then in order that $L_\rho \in \mathscr{S}_n(\mathscr{P})$ we must have

$$P(1 + x_i) = P_1(1 + x_i), \tag{3.4}$$

$$P(x_i) = P_{-1}(x_i), \qquad \forall i = 1, 2, \ldots, n - s + 1, \tag{3.5}$$

where we adopt the convention that the polynomials are replaced by their derivatives if x_i is a multiple zero of p, and if α and $(-1 + \alpha)$ are zeros of p of multiplicity m and l, respectively, the corresponding equations in (3.4) and (3.5) are

$$\begin{aligned} P^{(k)}(1 + \alpha) &= P_{-1}^{(k)}(\alpha), \qquad \forall k = 0, 1, \ldots, m \wedge s - 1, \quad k \neq \rho, \\ P^{(\rho)}(1 + \alpha) &= P_{-1}^{(\rho)}(\alpha) - 1, \end{aligned} \tag{3.6}$$

and

$$\begin{aligned} P^{(k)}(-1 + \alpha) &= P_1^{(k)}(\alpha), \qquad \forall k = 0, 1, \ldots, l \wedge s - 1, \quad k \neq \rho, \\ P^{(\rho)}(-1 + \alpha) &= P_1^{(\rho)}(\alpha) - 1. \end{aligned} \tag{3.7}$$

Then (3.4) and (3.5), with the corresponding equations replaced by (3.6) and (3.7) if p has zeros at α and $(-1 + \alpha)$, give a nonhomogenous system of $d + (n - s + 1)$ equations in $d + (n - s + 1)$ unknowns $c_1, c_2, \ldots, c_d$, $a_s, a_{s+1}, \ldots, a_n$. The corresponding homogenous system is obtained by replacing the polynomial P in (3.3) by one without the term $(x - \alpha)^\rho/\rho!$. If the system is singular it would mean that there exists a nonzero function $S \in \mathscr{S}_n{}^\alpha(\mathscr{P})$ which is bounded. This is impossible by Lemma 2.14. Hence the system is nonsingular, so that the function L_ρ is uniquely defined.

The spline function L_ρ $(\rho = 0, 1, \ldots, s - 1)$ has the following properties:

$$L_\rho^{(k)}(\nu) = 0, \qquad \forall \nu = \pm 1, \pm 2, \pm 3, \ldots \quad \text{and} \quad \forall k = 0, 1, \ldots, s - 1. \tag{3.8}$$

$$L_\rho^{(k)}(0) = \delta_{k\rho}, \qquad \forall k = 0, 1, \ldots, s - 1, \tag{3.9}$$

and $L_\rho(x) \to 0$ exponentially as $|x| \to \infty$.

Now define

$$S(x) = \sum_{\nu=-\infty}^{\infty} y_\nu L_0(x - \nu) + \sum_{\nu=-\infty}^{\infty} y_\nu^{(\nu)} L_1(x - \nu) + \cdots$$

$$+ \sum_{\nu=-\infty}^{\infty} y_\nu^{(s-1)} L_{s-1}(x - \nu). \tag{3.10}$$

Then $S \in \mathscr{S}_n(\mathscr{P})$ and satisfies (1.5) and (1.6) of Theorem 1.

If $S_1 \in \mathscr{S}_n(\mathscr{P})$ also satisfies (1.5) and (1.6) then $S - S_1 \in \mathscr{S}_n^{\alpha}(\mathscr{P})$ and is of power growth as $| x | \to \infty$. By Lemma 2.14, then, we must have $S \equiv S_1$. ∎

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